Geometric Transformations of the Complex Plane

Aryan Raj Dhawan

May 7, 2026

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1 Introduction

There are several different classes of analytic functions, each with their own distinct and interesting properties. In particular, we can view analytic functions as mappings from one space to another, which leads to many useful geometric interpretations as well.

To this end, we first explore conformal mappings, which preserve the angle between two smooth paths. After rigorously defining such mappings and the idea of angle preservation, we then demonstrate its application by an overview of the complex exponential function as well as stereographic projection.

The discussion on stereographic projection as a conformal mapping offers a fitting segue into the discussion of primitive mappings such as translations, rotations, reflections, dilations, and inversions. After exploring these, we delve deeper into classes of transformations that arise from compositions of the primitive mappings. We start with linear transformations, which map geometrically similar objects to one another. Then, we conclude with linear fractional transformations and Mobius transformations which have even higher expressive power in terms of the transformations they can be used to represent. We again begin with an exploration of the more algebraic aspects of such transformations, including their matrix correspondence from a group theory perspective. Then, we depict both their use in depicting any combination of the primitive transformations as well as portray how action on three distinct points uniquely defines such a transformation.

2 Background

To precede our discussion of conformal mappings, we must first define what it means for a complex-valued function to be analytic as well as formalize the idea of the angle between intersecting paths on the Argand plane.

Definition 2.1. *Given an open set $G \subseteq \mathbb{C}$, a function $f : G \to \mathbb{C}$ is analytic if it is continuously differentiable on G, i.e for all $a \in G$ it is both continuous and differentiable at a.*

Definition 2.2. *Consider two smooth paths γ_1, γ_2 in $G \subseteq \mathbb{C}$ that intersect at $z_0 = \gamma_1(t_1) = \gamma_2(t_2)$ such that $\gamma_1'(t_1) \neq 0 \neq \gamma_2'(t_2)$. The angle between γ_1, γ_2 at z_0 is defined to be*

$$\theta = Arg(\gamma_2'(t_2)) - Arg(\gamma_1'(t_1))$$

where $\operatorname{Arg}(z) \in (-\pi, \pi]$ denotes the principal argument of $z \in \mathbb{C}$. The difference is understood modulo 2π, since the difference of two principal arguments need not itself lie in $(-\pi, \pi]$.

The intuition behind this definition can be seen in the picture below. At the point $z_0 \in G$, we consider the tangent line to $\gamma_1(t_1)$. The tangent direction at z_0 is given by the vector $\gamma_1'(t_1)$, whose argument $\arg(\gamma_1'(t_1))$ represents the angle that the tangent line makes with the positive real axis. Similarly, the

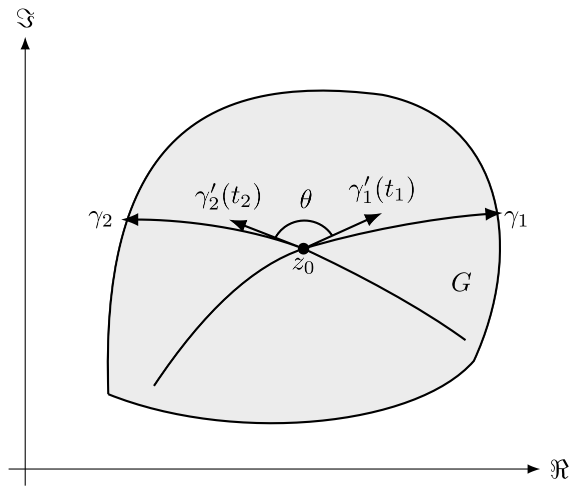

Figure 1: In the z-plane and w-plane, the angle between two smooth curves γ_1, γ_2 with non-vanishing tangents at their point of intersection z_0 is the angle between their tangent lines at z_0

tangent direction at z_0 along γ_2 is given by $\gamma_2'(t_2)$, and its argument $\arg(\gamma_2'(t_2))$ represents the corresponding angle with the positive real axis. Note that the additional assumption of non-vanishing first derivatives ensure that these tangent directions are well-defined. Therefore, the angle between the two paths at z_0 is naturally defined as the difference of the aforementioned angles.

3 Conformal Maps

3.1 Conformality and Analyticity

Definition 3.1. *A conformal map is a function $f : G \to \mathbb{C}$, where $G \subseteq \mathbb{C}$ is open and connected such that f preserves angles at all $a \in G$ and the following limit exists.*

$$\lim_{z \to a} \frac{|f(z) - f(a)|}{|z - a|}$$

We can already observe a connection between conformal mappings and analytic functions, owing to the similarity between the aforementioned limit and the definition fo the complex derivative. Before we outline this connection explicitly, we analyze the angle preserving properties of analytic functions.

Lemma 3.2. *If $\gamma : [a, b] \to G$ is smooth and $f : G \to \mathbb{C}$ is analytic, then the composition $\sigma = f \circ \gamma$ is also a smooth path and $\sigma'(t) = f'(\gamma(t))\gamma'(t)$.*

Proof. Since γ is a path it is continuous by definition, and f is continuous as it is analytic. Hence, $\sigma = f \circ \gamma$, the composition of continuous functions, is continuous. Thus, σ is a path.

To prove that it's smooth, we note that $\gamma'(t)$ exists for all $t \in [a, b]$ since γ is smooth. Further, f is analytic so $f'(z)$ exists for all $z \in G$. Hence, $\sigma'(t)$ is well-defined for all $t \in [a, b]$, and by the chain rule we have $\sigma'(t) = f'(\gamma(t))\gamma'(t)$. $\square$

Lemma 3.3. *For $z_1, z_2 \in \mathbb{C}$, $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$ modulo 2π.*

Proof. Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$. So, $\theta_1 = \arg(z_1), \theta_2 = \arg(z_2)$. We then have $z_1 z_2 = r_1 r_2 e^{i\theta_1 + i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$. Hence, $\arg(z_1 z_2) = \theta_1 + \theta_2 = \arg(z_1) + \arg(z_2)$ modulo 2π. $\square$

Remark 3.4. *The equality holds without qualification for the multi-valued* arg*. For the principal argument* Arg*, it holds only modulo 2π (e.g.,* $\operatorname{Arg}(-1) + \operatorname{Arg}(i) = \frac{3\pi}{2}$*, while* $\operatorname{Arg}(-i) = -\frac{\pi}{2}$*). This is sufficient for our purposes, since the angle between curves in Definition 2.2 is itself defined only up to integer multiples of 2π.*

Theorem 3.5. *If $f : G \to \mathbb{C}$ is analytic and $f'(z_0) \neq 0$ for some $z_0 \in G$, then f preserves angles at z_0.*

Proof. We show that for any two smooth paths $\gamma_1, \gamma_2 : [a, b] \to G$ such that $\gamma_1(t_1) = \gamma_2(t_2) = z_0 \in G$ (thus intersecting at z_0) and $\gamma_1'(t_1) \neq 0 \neq \gamma_2'(t_2)$ (non-vanishing tangent), the angle between γ_1 and γ_2 at z_0 is equal to the angle between $\sigma_1 = f \circ \gamma_1$ and $\sigma_2 = f \circ \gamma_2$ at $\omega_0 = f(z_0)$.

We begin by also noting that $f'(z_0) \neq 0$ by assumption. From Lemma 1, since γ_1 is smooth and f is analytic, we have $\sigma_1'(t_1) = f'(\gamma_1(t_1))\gamma_1'(t_1) = f'(z_0)\gamma_1'(t_1)$. As both $f'(z_0), \gamma_1'(t_1)$ are non-zero, $\sigma_1'(t_1) \neq 0$. Further, taking the argument of both sides, we obtain $\arg(\sigma_1'(t_1)) = arg(f'(z_0)) + arg(\gamma_1'(t_1))$ by Lemma 2. Rearranging this equation, we have

$$arg(f'(z_0)) = \arg(\sigma_1'(t_1)) - arg(\gamma_1'(t_1))$$

Without loss of generality, since γ_2 is also a smooth path in G with $z_0 = \gamma_2(t_2)$, it follows that

$$arg(f'(z_0)) = \arg(\sigma_2'(t_2)) - arg(\gamma_2'(t_2))$$

Therefore,

$$\arg(\sigma_1'(t_1)) - arg(\gamma_1'(t_1)) = \arg(\sigma_2'(t_2)) - arg(\gamma_2'(t_2))$$

$$\implies arg(\gamma_2'(t_2)) - arg(\gamma_1'(t_1)) = \arg(\sigma_2'(t_2)) - \arg(\sigma_1'(t_1))$$

Hence, the angle between γ_1, γ_2 at z_0, the left side of this equation, equals the angle between σ_1, σ_2 at $\omega_0 = f(z_0)$

$\square$

We can, in fact, make a much stronger statement vis-a-vis the relationship between conformal maps and analytic functions.

Theorem 3.6. *If $G \subseteq C$ is open, $f : G \to \mathbb{C}$ is conformal if and only if f is analytic and for all $z_0 \in G$ $f'(z_0) \neq 0$*

Proof. If f is analytic and for all $z_0 \in G$ we have $f'(z_0) \neq 0$ then, by Theorem 1, it follows that f preserves angles at all $z_0 \in G$. Further, let $z_0 \in G$. Observe that

$$\lim_{z \to z_0} \frac{|f(z) - f(z_0)|}{|z - z_0|} = \lim_{z \to z_0} \left| \frac{f(z) - f(z_0)}{z - z_0} \right| = \left| \lim_{z \to a} \frac{f(z) - f(z_0)}{z - z_0} \right| = |f'(z_0)|$$

which exists since f is analytic. Thus, f is conformal by definition.

Conversely, suppose that f is conformal on G, and assume that the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ are continuous. Let $z_0 \in G$, and we then consider a smooth path

$$\gamma(t) = x(t) + iy(t)$$

with $\gamma(t_0) = z_0$ and $\gamma'(t_0) \neq 0$. Let

$$w(t) = f(\gamma(t)) \implies w'(t) = f'(\gamma(t))\gamma'(t)$$

Since the partial derivatives of f are continuous, we can use the chain rule to give us an equivalent expression as so,

$$w'(t_0) = \frac{\partial f}{\partial x}\, x'(t_0) + \frac{\partial f}{\partial y}\, y'(t_0),$$

where the partial derivatives are evaluated at z_0.

Writing this in terms of $\gamma'(t_0)$, we obtain

$$w'(t_0) = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)\gamma'(t_0) + \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)\overline{\gamma'(t_0)}.$$

Dividing by $\gamma'(t_0) \neq 0$ yields

$$\frac{w'(t_0)}{\gamma'(t_0)} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) + \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)\frac{\overline{\gamma'(t_0)}}{\gamma'(t_0)}.$$

Since f preserves angles, we note that the quantity

$$\arg\left(\frac{w'(t_0)}{\gamma'(t_0)}\right)$$

must be independent of the direction of the tangent vector $\gamma'(t_0)$. However,

$$\left|\frac{\overline{\gamma'(t_0)}}{\gamma'(t_0)}\right| = 1$$

since $|\bar{w}| = |w|$ for any $w \in \mathbb{C}$, and as the direction of $\gamma'(t_0)$ varies this term traces out the unit circle [4]. Thus the expression above describes a circle in the complex plane. Its argument can be independent of the direction only if this circle has radius zero. Hence

$$\frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = 0,$$

which implies

$$\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}.$$

Thus, the Cauchy–Riemann equations are satisfied (and f_x, f_y are continuous). Therefore, f is analytic at z_0. Since z_0 was arbitrary, f is analytic on G.

Finally, from the expression for $w'(t_0)$ we obtain

$$w'(t_0) = \frac{\partial f}{\partial x}\,\gamma'(t_0),$$

so

$$\frac{w'(t_0)}{\gamma'(t_0)} = \frac{\partial f}{\partial x} = f'(z_0).$$

If $f'(z_0) = 0$, then $w'(t_0) = 0$ for every smooth path through z_0, so the image curves would have zero tangent at $f(z_0)$ and the angle between them would not be defined. This contradicts the assumption that f preserves angles. Hence

$$f'(z_0) \neq 0.$$

Therefore f is analytic on G and $f'(z) \neq 0$ for all $z \in G$.

□

3.2 The Exponential Function

As an example, we now look at the complex exponential function $f(z) = e^z$. First, we show that f is conformal on $\mathbb{C}$.

Since we know f is analytic on $\mathbb{C}$ and $f'(z) = f(z) = e^z$, it remains to show that for all $z_0 \in \mathbb{C}$, $f'(z_0) \neq 0$. Suppose for the sake of contradiction, $f'(z_0) = e^{z_0} = 0$ for some $z_0 = x_0 + iy_0 \in \mathbb{C}$. Then

$$|e^{z_0}| = 0 \implies e^{x_0} = 0$$

which is a contradiction as 0 is not in the range of the real exponential function. Hence, by Theorem 2, f is conformal on $\mathbb{C}$.

We can consider two paths in $G = \mathbb{C}$, where $t \in [0, \pi]$

$$\gamma_1(t) = c + it$$

$$\gamma_2(t) = t + di$$

As can be seen, γ_1 represents a horizontal line segment whereas γ_2 represents a vertical line segment in the Argand plane. We note that γ_1, γ_2 intersect at

$z_0 = c + di$ when $t_1 = d, t_2 = c$. Further, we observe that the angle between the two paths is $\frac{\pi}{2}$. Since f is conformal, we expect this to be the angle between the images as well. Computing the image of γ_1,

$$\sigma_1(t) = f(\gamma_1(t)) = \left\{e^{c+it} : t \in [0, \pi]\right\} = \left\{e^c e^{it} : t \in [0, \pi]\right\}$$

which represents the semi-circle with radius $r = e^c$ in the first two quadrants. Similarly,

$$\sigma_2(t) = f(\gamma_2(t)) = \{e^t e^{di} : t \in [0, \pi]\}$$

which represents a ray at angle of d radians from the positive x-axis, with $0 < r = e^t < \pi$. As can be seen in the figure below, the angle between $\sigma_1(t), \sigma_2(t)$ at z_0 is $\frac{\pi}{2}$ as the radius of a circle is perpendicular to the tangent at any given point. A simple computation below shows this

$$\sigma_1'(t_1) = \sigma_1'(d) = ie^c e^{it_1} = e^c e^{i(t_1 + \frac{\pi}{2})} \to \arg(\sigma_1'(t)) = t_1 + \frac{\pi}{2} = d + \frac{\pi}{2}$$

$$\sigma_2'(t_2) = e^{t_2} e^{di} \to \arg(\sigma_2'(t_2)) = d$$

The angle between the two image paths is the difference of the arguments which is $\frac{\pi}{2}$. More importantly, this result matches the angle between the domain paths $\gamma_1(t), \gamma_2(t)$ which arises directly as a result of f being conformal.

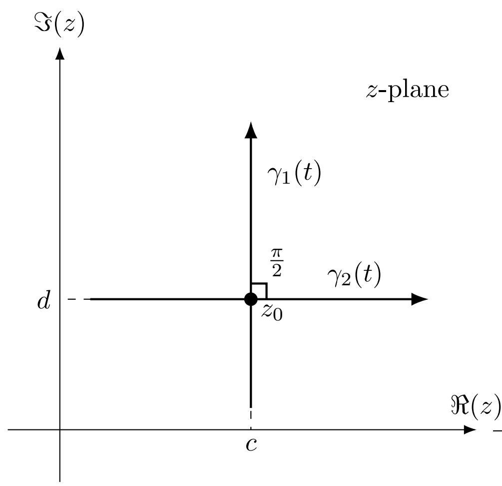

Figure 2: In the z-plane, the paths $\gamma_1(t) = c + it$ and $\gamma_2(t) = t + di$ intersect orthogonally at $z_0 = c + di$.

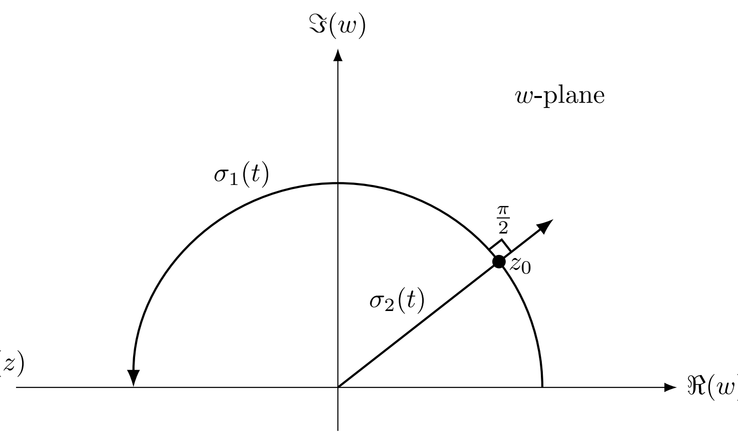

Figure 3: In the w-plane under $f(z) = e^z$, the vertical line maps to a semicircle and the horizontal line maps to a ray. The right angle is preserved.

3.3 Stereographic Projection

We now look at stereographic projection, which can also be viewed as a mapping from the unit sphere in $\mathbb{R}^3$ unit sphere to the complex plane. It provides a natural geometric motivation for the *extended complex plane* $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$, which will play a central role in our discussion of linear fractional transformations in Section 4.

Definition 3.7. *Let $S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$ denote the unit sphere, and let $N = (0, 0, 1)$ be the north pole.* ***Stereographic projection*** *is the map $\Pi : S^2 \to \mathbb{C}_\infty$ defined by drawing a line from N through a point $P = (x_1, x_2, x_3) \in S^2 \setminus \{N\}$ and taking its intersection with the plane $x_3 = 0$, identified with $\mathbb{C}$, and setting $\Pi(N) = \infty$. Explicitly,*

$$\Pi(x_1, x_2, x_3) = \begin{cases} \dfrac{x_1 + ix_2}{1 - x_3} & (x_1, x_2, x_3) \neq N \\ \infty & (x_1, x_2, x_3) = N \end{cases}$$

The inverse map sends $z = x + iy \in \mathbb{C}$ back to the sphere via

$$\Pi^{-1}(z) = \left(\frac{2x}{|z|^2 + 1}, \frac{2y}{|z|^2 + 1}, \frac{|z|^2 - 1}{|z|^2 + 1} \right)$$

and $\Pi^{-1}(\infty) = N$.

Since S^2 is compact and $\mathbb{C}$ is not, there can be no homeomorphism between them [5]. Thus restricting to $S^2 \setminus \{N\}$ (removing one point) is necessary to obtain a bijection onto $\mathbb{C}$. This is precisely why $\mathbb{C}$ alone is insufficient as a codomain, and motivates the introduction of $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$, which restores the "missing" point corresponding to N.

Theorem 3.8. *Stereographic projection $\Pi : S^2 \to \mathbb{C}_\infty$ is a bijection.*

Proof. We show Π is injective and surjective.

For injectivity, suppose $\Pi(x_1, x_2, x_3) = \Pi(x_1', x_2', x_3')$. If both points equal N, they are trivially equal. If exactly one equals N, then one image is ∞ and the other is finite, yielding a contradiction. Otherwise, neither is N and

$$\frac{x_1 + ix_2}{1 - x_3} = \frac{x_1' + ix_2'}{1 - x_3'}$$

Equating real and imaginary parts yields

$$x_1(1 - x_3') = x_1'(1 - x_3) \qquad \text{and} \qquad x_2(1 - x_3') = x_2'(1 - x_3)$$

We square and add these two equations,

$$x_1^2(1 - x_3')^2 + x_2^2(1 - x_3')^2 = x_1'^2(1 - x_3)^2 + x_2'^2(1 - x_3)^2$$

$$(x_1^2 + x_2^2)(1 - x_3')^2 = (x_1'^2 + x_2'^2)(1 - x_3)^2$$

Since both points lie on S^2, we have $x_1^2+x_2^2+x_3^2=1$ and $x_1'^2+x_2'^2+x_3'^2=1$, so $x_1^2+x_2^2=1-x_3^2=(1-x_3)(1+x_3)$ and similarly $x_1'^2+x_2'^2=(1-x_3')(1+x_3')$. Therefore,

$$(1-x_3)(1+x_3)(1-x_3')^2=(1-x_3')(1+x_3')(1-x_3)^2$$

Since neither point is N, we have $1-x_3\neq 0$ and $1-x_3'\neq 0$, so we may divide both sides by $(1-x_3)(1-x_3')$ to obtain

$$(1+x_3)(1-x_3')=(1+x_3')(1-x_3)$$

Expanding we obtain,

$$1-x_3'+x_3-x_3x_3'=1-x_3+x_3'-x_3x_3'$$

$$x_3-x_3'=x_3'-x_3 \implies 2x_3=2x_3' \implies x_3=x_3'$$

Substituting $x_3=x_3'$ back into the equations for the real and imaginary parts, we get $x_1(1-x_3)=x_1'(1-x_3)$ and $x_2(1-x_3)=x_2'(1-x_3)$. Since $1-x_3\neq 0$, we may divide to conclude $x_1=x_1'$ and $x_2=x_2'$. Hence $(x_1,x_2,x_3)=(x_1',x_2',x_3')$, and Π is injective.

For surjectivity, we must show every element of $\mathbb{C}_\infty$ has a preimage in S^2. We consider two cases.

If $w=\infty$, then $\Pi(N)=\infty$ by definition, so $N\in S^2$ is a preimage.

If $w=z=x+iy\in\mathbb{C}$, we claim the preimage is

$$\Pi^{-1}(z)=\left(\frac{2x}{|z|^2+1},\ \frac{2y}{|z|^2+1},\ \frac{|z|^2-1}{|z|^2+1}\right)$$

We first verify this point lies on S^2. Setting $r^2=|z|^2=x^2+y^2$ for convenience, we compute

$$\begin{aligned}\left(\frac{2x}{r^2+1}\right)^2+\left(\frac{2y}{r^2+1}\right)^2+\left(\frac{r^2-1}{r^2+1}\right)^2 &= \frac{4x^2+4y^2+(r^2-1)^2}{(r^2+1)^2}\\ &= \frac{4r^2+r^4-2r^2+1}{(r^2+1)^2}\\ &= \frac{r^4+2r^2+1}{(r^2+1)^2}\\ &= \frac{(r^2+1)^2}{(r^2+1)^2}=1\end{aligned}$$

so indeed $\Pi^{-1}(z)\in S^2$. Further, the third coordinate $\frac{r^2-1}{r^2+1}=1$ would require $r^2-1=r^2+1$, which is impossible, so $\Pi^{-1}(z)\neq N$. It remains to verify that

$\Pi(\Pi^{-1}(z)) = z$. Computing directly,

$$\begin{aligned}
\Pi\left(\frac{2x}{r^2+1}, \frac{2y}{r^2+1}, \frac{r^2-1}{r^2+1}\right) &= \frac{\dfrac{2x}{r^2+1} + i\dfrac{2y}{r^2+1}}{1 - \dfrac{r^2-1}{r^2+1}} \\
&= \frac{\dfrac{2x+2iy}{r^2+1}}{\dfrac{r^2+1-r^2+1}{r^2+1}} \\
&= \frac{\dfrac{2(x+iy)}{r^2+1}}{\dfrac{2}{r^2+1}} \\
&= x + iy = z
\end{aligned}$$

Hence every $z \in \mathbb{C}$ has a preimage in $S^2 \setminus \{N\}$, and together with the case $w = \infty$, we can conclude that Π is surjective onto $\mathbb{C}_\infty$. $\square$

Theorem 3.9. *Stereographic projection* Π *is conformal at every point of* $S^2 \setminus \{N\}$*, and is not conformal at* N*.*

Proof. Conformality at points of $S^2 \setminus \{N\}$ requires more sophisticated concepts of tangent spaces and differential geometry, which is beyond the scope of this paper. We refer the reader to [1] for a complete proof.

For non-conformality at N: by Definition 3.1, conformality requires the limit

$$\lim_{P \to N} \frac{|\Pi(P) - \Pi(N)|}{d_{S^2}(P, N)}$$

to exist finitely. Since $\Pi(N) = \infty$, no such finite limit exists, and hence Π is not conformal at N. Note that in this case, the domain is S^2 (instead of $\mathbb{C}$ specified in the definition) and thus we use the distance metric associated with S^2. $\square$

As we will see, the extended complex plane $\mathbb{C}_\infty$ is the natural domain for linear fractional transformations as well. A Möbius transformation has a pole at some $z_0 \in \mathbb{C}$ and is therefore most naturally viewed as a bijection $\mathbb{C}_\infty \to \mathbb{C}_\infty$, as we shall see in Section 4. The one-point compactification of $\mathbb{C}$ by ∞, made geometric here via stereographic projection, is therefore more than a mere notational convenience.

4 Linear Fractional Transformations

4.1 Primitive Mappings and Isometries

Definition 4.1. *An **isometry** of $\mathbb{C}$ is a function $f : \mathbb{C} \to \mathbb{C}$ such that for all $z_1, z_2 \in \mathbb{C}$,*

$$|z_2 - z_1| = |f(z_2) - f(z_1)|$$

We again return to the notion of mappings preserving some geometric quantity. Whereas conformal mappings preserve angles between curves, isometries are mappings that preserve the distance between any two points. We now explore some fundamentals isometries of $\mathbb{C}$: translations, rotations, and reflections.

Definition 4.2. *A **translation** is a function $f : \mathbb{C} \to \mathbb{C}$ defined by*

$$f(z) = z + c$$

for some $c \in \mathbb{C}$.

Clearly, this represents shifting ever point of the Argand plane in the direction of the vector corresponding to c.

Definition 4.3. *A **rotation** is a function $f : \mathbb{C} \to \mathbb{C}$ defined by*

$$f(z) = e^{i\theta} z$$

for some $\theta \in \mathbb{R}$.

This represents rotating every point of the plane, about the origin, by angle θ.

Definition 4.4. *A **reflection** is a function $f : \mathbb{C} \to \mathbb{C}$ defined by*

$$f(z) = \bar{z}$$

Observe that this represents reflection across the x-axis as it negates $\Im(z)$. Translations, rotations, and reflections are interesting since they preserve both distances and angles. We proceed to prove this below.

Theorem 4.5. *Translations, rotations, and reflections are isometries of $\mathbb{C}$. Further, translations and rotations are conformal mappings.*

Proof. Let $z_1, z_2 \in \mathbb{C}$.

Consider a translation f defined by $f(z) = z + c$ for $c \in \mathbb{C}$. It follows that

$$|f(z_2) - f(z_1)| = |(z_2 + c) - (z_1 + c)| = |z_2 - z_1|$$

Now, consider a rotation g defined by $g(z) = e^{i\theta} z$ for $\theta \in \mathbb{R}$.

$$|g(z_2) - g(z_1)| = |e^{i\theta} z_2 - e^{i\theta} z_1| = |e^{i\theta}(z_2 - z_1)|$$

$$= |e^{i\theta}||z_2 - z_1| = |z_2 - z_1|$$

Lastly, we consider a reflection h defined by $h(z) = \bar{z}$. Note that h (conjugation) is a homomorphism, i.e $h(z_1 + z_2) = h(z_1) + h(z_2)$ [2]. Further, if we write $z = x + iy$, then $|z| = \sqrt{x^2 + y^2} = \sqrt{x^2 + (-y)^2} = |\bar{z}|$. Thus, we obtain the following.

$$|h(z_2) - h(z_1)| = |\bar{z_2} - \bar{z_1}| = |\overline{z_2 - z_1}| = |z_2 - z_1|$$

Hence f, g, and h are all isometries of $\mathbb{C}$. It is also clear to see that both f and g are analytic. Further,

$$f'(z) = 1 \neq 0$$

$$g'(z) = e^{i\theta} \neq 0 \ \ \forall \theta$$

Hence, by Theorem 2, f and g are conformal.

Analyzing h further, we write $h(x + iy) = x - iy$ and thus we have $u = x,\ v = -y$. Hence,

$$\frac{\partial u}{\partial x} = 1 \neq -1 = \frac{\partial v}{\partial y}$$

The Cauchy-Riemann equations require $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, which fails everywhere. Hence h is nowhere analytic, and therefore not conformal by Theorem 2.

□

We proceed to define two more functions that are key to understanding linear fractional transformations.

Definition 4.6. *A **dilation** is a function $f : \mathbb{C} \to \mathbb{C}$ defined by*

$$f(z) = \rho z$$

for some $\rho \in \mathbb{R}$.

Again, if we view $z \in \mathbb{C}$ as a vector, multiplying by a real constant yields scaling z by the specified factor. By extension, dilations scale distances between two points $z_1, z_2 \in \mathbb{C}$ as well by the same scale factor ρ.

$$|f(z_2) - f(z_1)| = |\rho z_2 - \rho z_1| = |\rho(z_2 - z_1)| = \rho|z_2 - z_1|$$

Thus, clearly, when $\rho \neq 1$ a dilation is not an isometry.

Definition 4.7. *An **inversion** is a function $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ defined by*

$$f(z) = \frac{1}{z}$$

Inversions are particularly interesting, and can be viewed in regards to their effect on points inside the unit circle. Consider z_0 such that $0 < |z_0| < 1$ in the unit circle. Then $|f(z_0)| = \frac{1}{|z_0|} > 1$. Similarly, if $|z_0| > 1$, then $|f(z_0)| < 1$. Thus, all points inside the unit circle are mapped to the exterior and vice-versa. In this way, it turns the unit circle inside out [2].

4.2 Linear Transformations

We now begin to consider compositions of the primitive geometric transformations considered in section 4.1. The simplest form to initiate our exploration are linear transformations.

Definition 4.8. *A* ***linear transformation*** *is a function $f : \mathbb{C} \to \mathbb{C}$ defined by*

$$f(z) = az + b$$

where $a, b \in \mathbb{C}$ and $a \neq 0$

Theorem 4.9. *Every linear transformation is a composition of a rotation, dilation, and translation.*

Proof. Let $f(z) = az + b$ be a linear transformation. We write $a \in \mathbb{C}$ as $a = |a|e^{i\theta}$ for $\theta \in R$. Define $f_1(z) = e^{i\theta}z$, be a rotation where $\theta = \arg(a)$. Let $f_2(z) = |a|z$ be a dilation with scale factor $|a|$. Lastly, let $f_3(z) = z + b$ be a translation by b units. Then,

$$f_3 \circ f_2 \circ f_1(z) = f_3(f_2(e^{i\theta}z)) = f_3(|a|e^{i\theta}z) = f_3(az) = az + b = f(z)$$

Hence, $f = f_3 \circ f_2 \circ f_1$, the composition of a rotation, dilation, and translation. □

Thus, the image of any object under a linear transformation is geometrically similar to the original (and merely represents a scaled version with a potentially different position and orientation). We can see this in the following example as well.

Suppose we wish to map a circle $C_1 : |z - 1| = 1$ to a circle $C_2 : |w - i| = 2$. Since they're similar geometrically and differ only in radius (scale) and center (translation), a linear transformation is apt. We apply these transformations in this order.

First, we translate by -1 to center C_1 at the origin.

$$w_1 = z - 1$$

Second, we scale (dialate) by a factor of 2, achieving the desired radius.

$$w_2 = 2w_1 = 2z - 2$$

Lastly, we translate by 1 unit in the imaginary axis to obtain the desired center of C_2.

$$w = w_2 + i = 2z - 2 + i$$

This clearly matches the structure of a linear transformation with $a = 2$ and $b = i - 2$, thereby showing how we can use linear transformations to map geometrically similar objects to one another. To conclude this section, we also note that the set of linear transformations form a group. For completeness, we prove this below.

Theorem 4.10. *The set of linear transformations $\mathcal{L}$ is a group under function composition.*

Proof. Let $f(z) = az+b, g(z) = cz+d$ be linear transformations. Thus, $a, c \neq 0$. First, we note that $f \circ g(z) = f(cz+d) = acz + ad + b$, which is a valid linear transformation since $ac \neq 0$ owing to both a, c being nonzero by assumption. We therefore have closure in $\mathcal{L}$. Second, the identity $\mathrm{id}(z) = z$ is clearly a linear transformation with $a = 1 \neq 0$. Thirdly, for $f(z) = az + b$, $f^{-1}(z) = \frac{1}{a}z - \frac{b}{a}$ is an inverse. Note that f^{-1} is well-defined as $a \neq 0$ by assumption. Further,

$$f^{-1} \circ f(z) = f^{-1}(az+b) = \frac{1}{a}(az+b) - \frac{b}{a} = z + \frac{b}{a} - \frac{b}{a} = z = \mathrm{id}(z)$$

which verifies our claim. Lastly, function composition is always associative, completing the proof. Thus, $(\mathcal{L}, \circ)$ is a group.

□

Note that $f(z) \in \mathcal{L}$ is clearly analytic (being a polynomial) and $f'(z) = a \neq 0$, which means all linear transformations are conformal by Theorem 2. We thus obtain a fairly expressive class of transformations, but observe that an inversion can't be expressed as a linear transformation. We need to add more expressive power to our transformation class, and we do so by introducing the idea of a linear fractional transformation.

4.3 Mobius Transformations – Algebraic Aspects

Definition 4.11. *A* ***linear fractional transformation*** *is a function $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ defined by*

$$w = f(z) = \frac{az+b}{cz+d}$$

where $a, b, c, d \in \mathbb{C}$.

Definition 4.12. *A linear fractional transformation of the aforementioned form with $ad \neq bc$ is a* ***Mobius Transformation****.*

Theorem 4.13. *A Möbius transformation, viewed as a map $\mathbb{C}_\infty \to \mathbb{C}_\infty$, is a bijection.*

Proof. Let $f(z) = \frac{az+b}{cz+d}$ with $ad - bc \neq 0$. We extend f to $\mathbb{C}_\infty$ by setting $f(-d/c) = \infty$ and $f(\infty) = a/c$ when $c \neq 0$, and $f(\infty) = \infty$ when $c = 0$.

For injectivity on the finite portion of the domain, suppose $f(z_1) = f(z_2)$ with $cz_1 + d, cz_2 + d \neq 0$. Then

$$\frac{az_1+b}{cz_1+d} = \frac{az_2+b}{cz_2+d}.$$

Cross-multiplying and expanding, we obtain

$$(az_1+b)(cz_2+d) = (az_2+b)(cz_1+d),$$

$$adz_1 + bcz_2 = adz_2 + bcz_1,$$
$$(ad - bc)(z_1 - z_2) = 0.$$

Since $ad - bc \neq 0$, we have $z_1 = z_2$. Combined with the convention at the pole and at ∞, f is injective on $\mathbb{C}_\infty$.

For surjectivity, given $w \in \mathbb{C}$ with $cw - a \neq 0$, the equation $f(z) = w$ rearranges to

$$z = \frac{dw - b}{-cw + a},$$

which is well-defined and yields a preimage. The remaining points $w = a/c$ and $w = \infty$ are realized as the images of ∞ and $-d/c$ respectively, by construction. Hence every $w \in \mathbb{C}_\infty$ has a preimage. (An independent verification of surjectivity follows from the explicit construction of $f^{-1} \in \mathcal{M}$ in Theorem 4.14 below.) $\square$

To preceede our discussion of the geometric properties of the Mobius Transformation, we analyze some of the algebraic properties that these functions possess.

Theorem 4.14. *Let $\mathcal{M}$ be the set of all Mobius transformations. Then, $\mathcal{M}$ is a group under function composition.*

Proof. Let $f(z) = \frac{az+b}{cz+d}$ and $g(z) = \frac{ez+f}{gz+h}$ be Möbius transformations. Thus, we have $ad - bc \neq 0$ and $eh - fg \neq 0$.

First, we compute

$$(f \circ g)(z) = f\left(\frac{ez+f}{gz+h}\right) = \frac{a\left(\frac{ez+f}{gz+h}\right) + b}{c\left(\frac{ez+f}{gz+h}\right) + d} = \frac{a(ez+f) + b(gz+h)}{c(ez+f) + d(gz+h)} = \frac{(ae+bg)z + (af+bh)}{(ce+dg)z + (cf+dh)}.$$

This has the form of a linear fractional transformation with coefficients $a' = ae + bg,\ b' = af + bh,\ c' = ce + dg,\ d' = cf + dh$. To confirm it is a Möbius transformation, we verify $a'd' - b'c' \neq 0$:

$$a'd' - b'c' = (ae+bg)(cf+dh) - (af+bh)(ce+dg) = (ad-bc)(eh-fg) \neq 0,$$

since both factors are nonzero by assumption. Hence $f \circ g \in \mathcal{M}$.

As for the identity, we consider the map $\mathrm{id}(z) = z = \frac{1 \cdot z + 0}{0 \cdot z + 1}$ which has coefficients $a = d = 1,\ b = c = 0$, giving $ad - bc = 1 \neq 0$. Thus, $\mathrm{id} \in \mathcal{M}$. Clearly $f \circ \mathrm{id} = \mathrm{id} \circ f = f$ for all $f \in \mathcal{M}$.

Given $f(z) = \frac{az+b}{cz+d}$ with $ad - bc \neq 0$, define

$$f^{-1}(z) = \frac{dz - b}{-cz + a}.$$

This is a linear fractional transformation with determinant $da - (-b)(-c) = ad - bc \neq 0$, so $f^{-1} \in \mathcal{M}$. It remains to verify that this is indeed an inverse.

$$(f^{-1} \circ f)(z) = \frac{d\left(\frac{az+b}{cz+d}\right) - b}{-c\left(\frac{az+b}{cz+d}\right) + a} = \frac{d(az+b) - b(cz+d)}{-c(az+b) + a(cz+d)} = \frac{(ad-bc)z}{ad-bc} = z.$$

Since f^{-1} is the left inverse for f, it is therefore an inverse [6].

Lastly, to prove associativity, we use the fact that function composition is associative in general [6].

Since $\mathcal{M}$ is closed under function composition, contains the identity, and contains an inverse for each element, $(\mathcal{M}, \circ)$ is a group. $\square$

Theorem 4.15. *There is a surjective group homomorphism $\phi : GL_2(\mathbb{C}) \to \mathcal{M}$ from the group of invertible 2×2 matrices over $\mathbb{C}$ onto the group of Möbius transformations.*

Proof. We define $\phi : GL_2(\mathbb{C}) \to \mathcal{M}$ by

$$\phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \frac{az+b}{cz+d}.$$

Consider

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad B = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \in GL_2(\mathbb{C}).$$

We compute the matrix product as below.

$$AB = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}.$$

Then,

$$\phi(AB) = \frac{(ae+bg)z + (af+bh)}{(ce+dg)z + (cf+dh)}.$$

On the other hand, we compute $\phi(A) \circ \phi(B)$:

$$(\phi(A) \circ \phi(B))(z) = \phi(A)\left(\frac{ez+f}{gz+h}\right) = \frac{a\left(\frac{ez+f}{gz+h}\right)+b}{c\left(\frac{ez+f}{gz+h}\right)+d} = \frac{a(ez+f)+b(gz+h)}{c(ez+f)+d(gz+h)} = \frac{(ae+bg)z+(af+bh)}{(ce+dg)z+(cf+dh)}.$$

Hence $\phi(AB) = \phi(A) \circ \phi(B)$, so ϕ is a group homomorphism.

It remains to show that $R(\phi) = \mathcal{M}$. Consider $w = f(z) = \frac{az+b}{cz+d} \in \mathcal{M}$. We have $ad - bc \neq 0$. Consider $W = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then, $\phi(W) = w$. Further, $det(W) = ad - bc \neq 0$, and thus $W \in GL_2(\mathbb{C})$ since it's invertible.

Hence, ϕ is a group homomorphism onto $\mathcal{M}$

$\square$

This result is particularly important since this correspondence between Mobius Transformations $\mathcal{M}$ and invertible 2×2 matrices in $GL_2(\mathbb{C})$ gives us key insights based on the fact that they behave somewhat similarly. In fact, sometimes the quantity $ad - bc$ is referred to as the determinant of the transformation itself.

4.4 Mobius Transformations – Geometric Aspects

We now look into some of the more functional and geometric aspects of Mobius Transformations, starting with their conformal nature on all of $\mathbb{C}$, barring its pole.

Theorem 4.16. *Any Mobius Transformation $f(z)$ has a pole of order 1, and is conformal at every point except its pole.*

Proof. Let $f(z)$ be a Mobius transformation as defined above. By the chain rule, we obtain

$$f'(z) = \frac{a(cz+d) - c(az+b)}{(cz+d)^2} = \frac{acz + ad - acz - bc}{(cz+d)^2}$$
$$= \frac{ad - bc}{(cz+d)^2}$$

which exists at all points except isolated singularity $z_0 = \frac{-d}{c}$. Observe that since $g(z) = ad - bc$ is constant, it is analytic in a punctured neighborhood of z_0. Further, since f is a Mobius transformation, $g(z_0) = ad - bc \neq 0$. Thus, $f'(z)$ has a pole of order 2 at z_0 [2, Thm. 5.6].

To see that f itself has a pole of order 1 at z_0, observe that $cz+d = c(z-z_0)$ has a simple zero at z_0, while the numerator at z_0 evaluates to $a(-d/c)+b = (bc-ad)/c \neq 0$ (since $ad \neq bc$). Hence the numerator is nonzero at z_0 and the denominator has a simple zero, so f has a pole of order exactly 1 at z_0.

Since $f'(z)$ is analytic at all other points and $f'(z) \neq 0$ (as $ad-bc$ is nonzero), f is conformal away from its pole. □

Having established the conformality of Mobius transformations, a geometric interpretation – akin to that of the isometries discussed in section 4.1 – would demonstrate their usefulness. In fact, any Mobius Transformation is the composition of some of the mappings hitherto discussed.

Theorem 4.17. *Let S be a Mobius Transformation. Then, S is the composition of translation, dilations, rotations, and the inversion (not necessarily all four).*

Proof. Suppose $S(z) = \frac{az+b}{cz+d}$ is a Mobius Transformation. We proceed with the proof by cases based on our value of c as before.

First, we consider the case where $c = 0$. Then, $S(z) = \frac{az+b}{d} = \frac{a}{d}z + \frac{b}{d}$. Thus, define

$$S_1(z) = \frac{a}{d}z \quad : \text{dilation with scale factor } \frac{a}{d}$$
$$S_2(z) = z + \frac{b}{d} \quad : \text{translation by } \frac{b}{d} \text{ units}$$

Then, $S = S_2 \circ S_1$.

Now, we consider the case where $c \neq 0$. Let $\lambda = \frac{bc-ad}{c^2} \in \mathbb{C}$. We can write $\lambda = |\lambda|e^{i\theta}$ for some $\theta \in \mathbb{R}$. We define the following:

$$S_1(z) = z + \frac{d}{c} \quad : \text{translation by } \tfrac{d}{c}$$

$$S_2(z) = \frac{1}{z} \quad : \text{inversion}$$

$$S_3(z) = |\lambda| z \quad : \text{dilation by real scale factor } |\lambda| = \left|\tfrac{bc-ad}{c^2}\right|$$

$$S_4(z) = e^{i\theta} z \quad : \text{rotation by angle } \theta = \arg\left(\tfrac{bc-ad}{c^2}\right)$$

$$S_5(z) = z + \frac{a}{c} \quad : \text{translation by } \tfrac{a}{c}$$

Composing them, we obtain the following:

$$S_1(z) = z + \frac{d}{c},$$

$$S_2(S_1(z)) = \frac{1}{z + \frac{d}{c}} = \frac{c}{cz + d},$$

$$S_3(S_2(S_1(z))) = |\lambda| \cdot \frac{c}{cz + d} = \frac{|\lambda| c}{cz + d},$$

$$S_4(S_3(S_2(S_1(z)))) = e^{i\theta} \cdot \frac{|\lambda| c}{cz + d} = \frac{|\lambda| e^{i\theta} c}{cz + d} = \frac{\lambda c}{cz + d} = \frac{bc - ad}{c(cz + d)},$$

$$S_5(S_4(S_3(S_2(S_1(z))))) = \frac{bc - ad}{c(cz + d)} + \frac{a}{c} = \frac{bc - ad + a(cz + d)}{c(cz + d)} = \frac{acz + bc}{c(cz + d)} = \frac{az + b}{cz + d}.$$

Thus, $S = S_5 \circ S_4 \circ S_3 \circ S_2 \circ S_1$.

In both cases, S is expressed as a finite composition of translations, dilations, rotations, and inversions.

□

In fact, we can make a stronger claim – the converse also holds true.

Corollary 4.18. *Any finite composition of translations, dilations, rotations, and inversions is a Mobius transformation.*

Proof. Note that $(\mathcal{M}, \circ)$ is a group, as proved in Theorem 4. Thus, it suffices to check that each primitive map lies in $\mathcal{M}$.

A translation $z + c$ for $c \in \mathbb{R}$ is a Mobius transformation. We can simply take $a = 1, b = c, c = 0, d = 1$, giving $ad - bc = 1 \neq 0$.

A dilation ρz for $\rho \in \mathbb{R} \setminus \{0\}$ is a Mobius transformation. Here, we can take $a = \rho, b = 0, c = 0, d = 1$, giving $ad - bc = \rho \neq 0$.

A rotation $e^{i\theta} z$ is a Mobius transformation as well. We let $a = e^{i\theta}, b = 0, c = 0, d = 1$, giving $ad - bc = e^{i\theta} \neq 0$.

Finally, the inversion $\frac{1}{z}$ is a Mobius transformation. We can finally set $a = 0, b = 1, c = 1, d = 0$, giving $ad - bc = -1 \neq 0$.

Since each primitive map lies in $\mathcal{M}$ and $(\mathcal{M}, \circ)$ is closed, any finite composition of translations, rotations, dilations, and inversions is again a Mobius transformation. □

Having seen the expressive power (for geometric transformations) of the almost-everywhere analytic Mobius transformation, we conclude with a powerful result: all it takes to define a unique Mobius transformation is its action on three points in $\mathbb{C}_\infty$ (the extended plane, as discussed in section 3.3). Since S has a pole (Theorem 6), it's reasonable to view it as a map from $\mathbb{C}_\infty$ to C_∞. To this end, we first introduce the idea of a mapping's fixed points.

Definition 4.19. *A* ***fixed point*** z_0 *of a transformation* $f : G \to \mathbb{C}$*, where* $G \subseteq C$*, is a point such that* $f(z_0) = z_0$*.*

Lemma 4.20. *Let* S *be a Mobius Transformation that is not the identity. Then,* S *has at most two fixed points.*

Proof. Suppose $S = \frac{az+b}{cz+d} \neq \text{id}$ is a Mobius Transformation. Then, $ad - bc \neq 0$. Suppose z_0 is a fixed point of S. Then,

$$S(z_0) = z_0 \implies \frac{az_0 + b}{cz_0 + d} = z_0$$

$$az_0 + b = cz_0^2 + dz_0 \implies cz_0^2 + (d-a)z_0 + b = 0$$

First, we consider the case where $c \neq 0$. Then,

$$z_0 = \frac{-(d-a) \pm \sqrt{(d-a)^2 - 4bc}}{2c}.$$

Since $a, b, c, d \in \mathbb{C}$, the discriminant $(d-a)^2 - 4bc$ is itself a complex number, so the labels "positive" or "negative" do not apply. By the fundamental theorem of algebra, the quadratic always has roots in $\mathbb{C}$: two distinct roots if the discriminant is nonzero, and a single root with multiplicity two if it is zero. Hence in this subcase S has at most two fixed points in $\mathbb{C}$. Moreover, $S(\infty) = a/c \in \mathbb{C}$, so ∞ is not fixed when $c \neq 0$, and the count above is the total over $\mathbb{C}_\infty$.

Next, we consider the case where $c = 0$. Then $S(z) = \frac{a}{d}z + \frac{b}{d}$ is a linear function, so $S(\infty) = \infty$, meaning ∞ is a fixed point. The equation for finite fixed points becomes

$$(d-a)z_0 + b = 0.$$

If $d - a \neq 0$, this yields exactly one finite fixed point $z_0 = -b/(d-a)$, plus ∞, giving two fixed points in total. If $d - a = 0$ and $b \neq 0$, no finite point is fixed and only ∞ is fixed, giving one fixed point in total. If $d - a = 0$ and $b = 0$, then $S(z) = z$, contradicting the assumption $S \neq \text{id}$.

Since we exhaustively covered all cases, S has at most two fixed points in $\mathbb{C}_\infty$. □

Theorem 4.21. *A Mobius Transformation* S *is uniquely determined by its action on any three given points in* $\mathbb{C}_\infty$

Proof. Let a, b, c be distinct points in $\mathbb{C}_\infty$. We then let $\alpha = S(a), \beta = S(b), \gamma = S(c)$ be their images under Mobius transformation S. Suppose there exists another transformation T with the same property. Then, $T^{-1}(S(a)) = a, T^{-1}(S(b)) =$

$b, T^{-1}(S(c)) = c$. By definition, this means that a, b, c are fixed points of $T^{-1} \circ S$. Since $T^{-1} \circ S$ has 3 fixed points, by the previous Lemma, we know S must be the identity transformation. Thus, $T^{-1} \circ S = \mathrm{id} \iff S = T$. Thus, a Mobius Transformation is uniquely determined by its actions on any three given distinct points in $\mathbb{C}_\infty$.

$\square$

5 Conclusion

In this paper we have analyzed the progression of increasingly expressive classes of geometric mappings on the complex plane, organized by the geometric quantities they preserve. Beginning with conformal mappings and their characterization as analytic functions with non-vanishing derivative (Theorem 3.5), we established the foundational connection between the analytic and geometric points of view. The complex exponential was a concrete example, while stereographic projection provided the geometric motivation for the extended plane $\mathbb{C}_\infty$.

We then narrowed our focus to the primitive isometries – translations, rotations, and reflections – and the closely related dilations and inversions. Linear transformations $z \mapsto az + b$ emerged as compositions of rotations, dilations, and translations, and were shown to map geometrically similar figures to one another. Möbius transformations, defined by the additional condition $ad - bc \neq 0$, generalized this picture in several directions simultaneously: as bijections of $\mathbb{C}_\infty$ and as finite compositions of all four primitive maps (Theorem 4.17).

Three results, in particular, articulate the depth of structure within this class of transformations. First, the group $\mathcal{M}$ of Möbius transformations is the homomorphic image of $GL_2(\mathbb{C})$ (Theorem 4.15), so that purely matrix computations encode geometric facts about the action of $\mathcal{M}$ on $\mathbb{C}_\infty$. The quantity $ad - bc$ acquires the role of a determinant in both senses. Second, every Möbius transformation decomposes into the four primitive maps. Third, a Möbius transformation is uniquely determined by its action on three distinct points of $\mathbb{C}_\infty$ (Theorem 4.21), which reduces the construction of the transformation down to a simpler problem.

Further exploration promises exciting revelations as well. The classification of Möbius transformations via their fixed-point structure (refining Lemma 4.20) is the starting point for the theory of hyperbolic geometry on the upper half-plane ($\mathbb{H}^2$) and the unit disk, where Möbius transformations preserving the relevant boundary act as isometries [3]. Conformal maps more generally are governed by the Riemann Mapping Theorem [4], which states that any simply connected proper open subset of $\mathbb{C}$ is conformally equivalent to the unit disk. Thus, the elementary geometrical ideas developed here – and their inextricable link with complex analysis – serves as the foundation for a much richer theory.

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